

# Congruence Property in Orbifold Theory

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## Abstract

Let  $V$  be a rational, selfdual,  $C_2$ -cofinite vertex operator algebra of CFT type, and  $G$  a finite automorphism group of  $V$ . It is proved that the kernel of the representation of the modular group on twisted conformal blocks associated to  $V$  and  $G$  is a congruence subgroup. In particular, the  $q$ -character of each irreducible twisted module is a modular function on the same congruence subgroup. In the case  $V$  is the Frenkel-Lepowsky-Meurman's moonshine vertex operator algebra and  $G$  is the monster simple group, the generalized McKay-Thompson series associated to any commuting pair in the monster group is a modular function.

## 1 Introduction

The modular invariance of trace functions in orbifold theory was established in [DLM4]. That is, the twisted conformal blocks associated to a rational,  $C_2$ -cofinite vertex operator algebra  $V$  and a finite automorphism group  $G$  afford a representation of the modular group  $SL(2, \mathbb{Z})$ . We prove in this paper that the kernel of this representation is a congruence subgroup of  $SL(2, \mathbb{Z})$ .

The modular invariance of trace functions associated to any rational and  $C_2$ -cofinite vertex operator algebra was established in [Z] although the modular invariance of the characters of the rational conformal field theory [C] in physics, and the characters of the minimal representations of the Virasoro algebra [R] and integrable highest weight representations for the affine Kac-Moody algebras [KP] in mathematics was known much earlier. It was conjectured by many people that the kernel of the representation of  $SL(2, \mathbb{Z})$  in [Z] is a congruence subgroup. This conjecture was recently settled down in [DLN].

According to a well known conjecture in orbifold theory,  $V^G$  is rational for any rational vertex operator algebra  $V$  and any finite automorphism group  $G$ . It is easy to show that the trace functions defined in [DLM4] are vectors in the twisted conformal blocks associated vertex operator algebra  $V^G$ . So it is natural to expect that the kernel of the representation of  $SL(2, \mathbb{Z})$  on the twisted conformal blocks associated to  $V$  and  $G$  is a congruence subgroup. Unfortunately, the rationality and the  $C_2$ -cofiniteness of  $V^G$  were proved only when  $G$  is solvable in [M], [CM]. But this result for solvable group is sufficient for the congruence subgroup property in general orbifold theory. The key observation is that a trace function in the orbifold theory only involves with a commuting pairing  $(g, h)$  in  $G$ . Replacing  $G$  by the subgroup  $H$  generated by  $g, h$ , we see that  $V^H$  is rational and  $C_2$ -cofinite. We can then apply the congruence subgroup property result in [DLN] to prove any trace function in the twisted conformal blocks of orbifold theory is modular

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on a congruence subgroup. In particular, the  $q$ -character  $\chi_{V^G}(\tau)$  of  $V^G$  is a modular function. This leads us to a conjecture which characterizes the rational vertex operator algebra by the modularity of its  $q$ -character. A successful proof of this conjecture would imply that  $V^G$  is rational for any finite automorphism group  $G$ .

One important application is the modularity of the generalized McKay-Thompson series  $Z(g, h, \tau)$  associated to the moonshine vertex operator algebra  $V^\natural$  and its full automorphism group - the monster simple group  $\mathbb{M}$  [FLM], [G]. Norton's generalized moonshine conjecture [N] says that for any commuting paring  $(g, h)$  in  $\mathbb{M}$  there is a genus zero modular function  $Z(g, h, \tau)$  such that (1) The coefficient of each power of  $q$  in  $Z(g, h, \tau)$  is a projective character of  $C_{\mathbb{M}}(g)$ , (2)  $Z(g, h, \tau)$  is invariant under the conjugation, (3) For any  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  in  $SL(2, \mathbb{Z})$ , there are nonzero constants  $\gamma_{(g, h), (g^a h^c, g^b h^d)}$  such that

$$Z(g, h, \gamma\tau) = \gamma_{(g, h), (g^a h^c, g^b h^d)} Z(g^a h^c, g^b h^d, \tau),$$

(4)  $Z(1, g, \tau)$  is the original McKay-Thompson series [CN]. It was pointed out in [DLM4] that the trace functions  $Z(g, h, \tau)$  appearing in the orbifold theory associated to  $V^\natural$  and  $\mathbb{M}$  satisfy conditions (1)-(4). As a corollary of the main result in this paper, we see that each  $Z(g, h, \tau)$  is a modular function over a congruence subgroup. But the genus zero property of  $Z(g, h, \tau)$  remains open except that the subgroup generated by  $g$  and  $h$  is cyclic [B], [DLM4].

The paper is organized as follows. Section 2 covers the twisted modules,  $g$ -rationality and related concepts following [DLM2], [DLM3]. We also present some important results for  $g$ -rational vertex operator algebras. Section 3 is a review of the modular invariance of trace functions in orbifold theory from [DLM4]. We prove the main result in Section 4. That is, the kernel of the representation of  $SL(2, \mathbb{Z})$  on the twisted conformal blocks in orbifold theory is a congruence subgroup. In the last section we consider the special case when the vertex operator algebra is holomorphic and we also discuss how these results are related to the generalized moonshine conjecture [N] for the moonshine vertex operator algebra  $V^\natural$  and the monster group  $\mathbb{M}$ .

## 2 Basics

In this section we recall various notions of twisted modules for a vertex operator algebra following [DLM3] and discuss some important concepts such as  $g$ -rationality, regularity, and  $C_2$ -cofiniteness from [Z] and [DLM2], [DLM3].

We first define the  $C_2$ -cofiniteness [Z].

**Definition 2.1.** *We say that a vertex operator algebra  $V$  is  $C_2$ -cofinite if  $V/C_2(V)$  is finite dimensional, where  $C_2(V) = \langle v_{-2}u | v, u \in V \rangle$ .*

Fix vertex operator algebra  $V$  and an automorphism  $g$  of  $V$  of finite order  $T$ . Decompose  $V$  into a direct sum of eigenspaces of  $g$  :

$$V = \bigoplus_{r \in \mathbb{Z}/T\mathbb{Z}} V^r,$$

where  $V^r = \{v \in V | gv = e^{-2\pi ir/T} v\}$ . We use  $r$  to denote both an integer between 0 and  $T - 1$  and its residue class modulo  $T$  in this situation.

**Definition 2.2.** A weak  $g$ -twisted  $V$ -module  $M$  is a vector space equipped with a linear map

$$Y_M : V \rightarrow (\text{End } M)[[z^{1/T}, z^{-1/T}]]$$

$$v \mapsto Y_M(v, z) = \sum_{n \in \frac{1}{T}\mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \text{End } M)$$

satisfying the following: for all  $0 \leq r \leq T - 1$ ,  $u \in V^r$ ,  $v \in V$ ,  $w \in M$ ,

$$Y_M(u, z) = \sum_{n \in \frac{r}{T} + \mathbb{Z}} u_n z^{-n-1},$$

$$u_l w = 0 \quad \text{for } l \gg 0,$$

$$Y_M(\mathbf{1}, z) = Id_M,$$

$$z_0^{-1} \delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(u, z_1) Y_M(v, z_2) - z_0^{-1} \delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(v, z_2) Y_M(u, z_1)$$

$$= z_2^{-1} \left(\frac{z_1 - z_0}{z_2}\right)^{-r/T} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(u, z_0)v, z_2),$$

where  $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$  and all binomial expressions are to be expanded in nonnegative integral powers of the second variable.

**Definition 2.3.** A  $g$ -twisted  $V$ -module is a  $\mathbb{C}$ -graded weak  $g$ -twisted  $V$ -module  $M$  :

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$$

where  $M_\lambda = \{w \in M | L(0)w = \lambda w\}$  and  $L(0)$  is the component operator of  $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2}$ . We also require that  $\dim M_\lambda$  is finite and for fixed  $\lambda$ ,  $M_{\frac{n}{T} + \lambda} = 0$  for all small enough integers  $n$ . If  $w \in M_\lambda$  we call  $\lambda$  the weight of  $w$  and write  $\lambda = \text{wt} w$ .

**Definition 2.4.** Let  $\mathbb{Z}_+$  be the set of nonnegative integers. An admissible  $g$ -twisted  $V$ -module is a  $\frac{1}{T}\mathbb{Z}_+$ -graded weak  $g$ -twisted  $V$ -module  $M$  :

$$M = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_+} M(n)$$

satisfying

$$v_m M(n) \subseteq M(n + \text{wt} v - m - 1)$$

for homogeneous  $v \in V$ ,  $m, n \in \frac{1}{T}\mathbb{Z}$ .

In the case  $g$  is the identity map, we have the notions of weak, ordinary and admissible  $V$ -modules [DLM3].

We also need the notion of the contragredient module. For an admissible  $g$ -twisted  $V$ -module  $M = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_+} M(n)$ , the contragredient module  $M'$  is defined as

$$M' = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_+} M(n)^*,$$

where  $M(n)^* = \text{Hom}_{\mathbb{C}}(M(n), \mathbb{C})$ . The vertex operator  $Y_{M'}(a, z)$  is defined for  $a \in V$  via

$$\langle Y_{M'}(a, z)f, u \rangle = \langle f, Y_M(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})u \rangle,$$

where  $\langle f, w \rangle = f(w)$  is the natural pairing  $M' \times M \rightarrow \mathbb{C}$ . Then  $(M', Y_{M'})$  is an admissible  $g^{-1}$ -twisted  $V$ -module [FHL], [X]. The contragredient module  $M'$  for a  $g$ -twisted  $V$ -module  $M$  can also be defined in the same fashion. In this case,  $M'$  is a  $g^{-1}$ -twisted  $V$ -module. Moreover,  $M$  is irreducible if and only if  $M'$  is irreducible.  $V$  is called selfdual if  $V$  and  $V'$  are isomorphic  $V$ -modules.

**Definition 2.5.** *A vertex operator algebra  $V$  is called  $g$ -rational, if the admissible  $g$ -twisted module category is semisimple.  $V$  is called rational if  $V$  is 1-rational where 1 is the identity map on  $V$ .*

The following results from [DLM3] tell us why rationality is important.

**Theorem 2.6.** *Assume that  $V$  is  $g$ -rational. Then*

- (1) *Any irreducible admissible  $g$ -twisted  $V$ -module  $M$  is a  $g$ -twisted  $V$ -module. Moreover, there exists a number  $\lambda \in \mathbb{C}$  such that  $M = \bigoplus_{n \in \frac{1}{T}\mathbb{Z}_+} M_{\lambda+n}$  where  $M_{\lambda} \neq 0$ . The  $\lambda$  is called the conformal weight of  $M$ .*
- (2) *There are only finitely many inequivalent irreducible admissible  $g$ -twisted  $V$ -modules.*
- (3) *If  $V$  is also  $C_2$ -cofinite and  $g^i$ -rational for all  $i \geq 0$  then the central charge  $c$  and the conformal weight  $\lambda$  of any irreducible  $g$ -twisted  $V$ -module  $M$  are rational numbers.*

A vertex operator algebra  $V$  is called regular if every weak  $V$ -module is a direct sum of irreducible  $V$ -modules [DLM2]. A vertex operator algebra  $V = \bigoplus_{n \in \mathbb{Z}} V_n$  is said to be of CFT type if  $V_n = 0$  for negative  $n$  and  $V_0 = \mathbb{C}\mathbf{1}$ . It is proved in [ABD], [L] that if  $V$  is of CFT type, then regularity is equivalent to rationality and  $C_2$ -cofiniteness. Also  $V$  is regular if and only if the weak module category is semisimple [DYu].

The following results from [M] and [CM] play important roles in this paper.

**Theorem 2.7.** *If  $V$  is a regular vertex operator algebra of CFT type and  $G$  is a finite solvable subgroup of  $\text{Aut}(V)$  then  $V^G$  is regular.*

Using a result from [ADJR] and Theorem 2.7 we have

**Theorem 2.8.** *If  $V$  is regular vertex operator algebra of CFT type then  $V$  is  $g$ -rational for any finite order automorphism  $g$  of  $V$ .*

### 3 Modular invariance

We present the modular invariance property of the trace functions in orbifold theory from [Z] and [DLM4] in this section.

In order to discuss the modular invariance of trace functions we consider the action of  $\text{Aut}(V)$  on twisted modules [DLM4]. Let  $g, h$  be two automorphisms of  $V$  with  $g$  of finite order. If  $(M, Y_M)$  is a weak  $g$ -twisted  $V$ -module, then  $(M \circ h, Y_{M \circ h})$  is a weak  $h^{-1}gh$ -twisted  $V$ -module where  $M \circ h \cong M$  as vector spaces and

$$Y_{M \circ h}(v, z) = Y_M(hv, z)$$

for  $v \in V$ . This defines a right action of  $\text{Aut}(V)$  on weak twisted  $V$ -modules and on isomorphism classes of weak twisted  $V$ -modules. For short, we write

$$(M, Y_M) \circ h = (M \circ h, Y_{M \circ h}) = M \circ h.$$

Assume that  $g, h$  commute. Denote by  $\mathcal{M}(g)$  the equivalence classes of irreducible  $g$ -twisted  $V$ -modules and set  $\mathcal{M}(g, h) = \{M \in \mathcal{M}(g) | h \circ M \cong M\}$ . Then  $\mathcal{M}(g, h)$  is a subset of  $\mathcal{M}(g)$ . By Theorems 2.6 and 2.8, if  $V$  is a regular vertex operator algebra of CFT type, both  $\mathcal{M}(g)$  and  $\mathcal{M}(g, h)$  are finite sets. Let  $M \in \mathcal{M}(g, h)$ . There is a linear isomorphism  $\varphi(h)$  from  $M$  to  $M$  such that :

$$\varphi(h)Y_M(v, z)\varphi(h)^{-1} = Y_M(hv, z)$$

for  $v \in V$ . Note that  $\varphi(h)$  is unique up to a nonzero scalar. If  $h = 1$  we simply take  $\varphi(1) = 1$ . For  $v \in V$  we set

$$Z_M(v, (g, h), \tau) = \text{tr}_M o(v)\varphi(h)q^{L(0)-c/24} = q^{\lambda-c/24} \sum_{n \in \frac{1}{T}\mathbb{Z}_+} \text{tr}_{M_{\lambda+n}} o(v)\varphi(h)q^n$$

which is a holomorphic function on the upper half plane  $\mathbb{H}$  [DLM4] with  $q = e^{2\pi i\tau}$ . Note that  $Z_M(v, (g, h), \tau)$  is defined up to a nonzero scalar. If  $h = 1$  and  $v = \mathbf{1}$  then  $Z_M(\mathbf{1}, (g, 1), \tau)$  is called the  $q$ -character of  $M$  and is denoted by  $\chi_M(\tau)$ .

For the modular invariance we need another vertex operator algebra  $(V, Y[\cdot], \mathbf{1}, \tilde{\omega})$  associated to  $V$  in [Z]. Here  $\tilde{\omega} = \omega - c/24$  and

$$Y[v, z] = Y(v, e^z - 1)e^{z \cdot \text{wt}v} = \sum_{n \in \mathbb{Z}} v[n]z^{-n-1}$$

for homogeneous  $v$ . We also write

$$Y[\tilde{\omega}, z] = \sum_{n \in \mathbb{Z}} L[n]z^{-n-2}.$$

If  $v \in V$  is homogeneous in the second vertex operator algebra, we denote its weight by  $\text{wt}[v]$ .

From now on we assume that  $V$  is a regular vertex operator algebra of CFT type and  $G$  a finite automorphism group of  $V$ . Set  $\mathcal{M} = \cup_{g \in G} \mathcal{M}(g)$ . As usual, let  $P(G)$  denote the

ordered commuting pairs in  $G$ . For  $(g, h) \in P(G)$  and  $M \in \mathcal{M}(g, h)$ ,  $Z_M(v, (g, h), \tau)$  is a function on  $V \times \mathbb{H}$ . Let  $\mathcal{W}$  be the vector space spanned by these functions. It is clear that the dimensional of  $\mathcal{W}$  is equal to  $\sum_{(g, h) \in P(G)} |\mathcal{M}(g, h)|$ . Define an action of the modular group  $\Gamma = SL(2, \mathbb{Z})$  on  $\mathcal{W}$  such that

$$Z_M|_\gamma(v, (g, h), \tau) = (c\tau + d)^{-\text{wt}[v]} Z_M(v, (g, h), \gamma\tau),$$

where

$$\gamma : \tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \quad (3.1)$$

We let  $\gamma \in \Gamma$  act on the right of  $P(G)$  via

$$(g, h)\gamma = (g^a h^c, g^b h^d).$$

Using Theorem 2.8 we have the following results [DLM4] .

**Theorem 3.1.** (1) *There is a representation  $\rho : \Gamma \rightarrow GL(\mathcal{W})$  such that for  $(g, h) \in P(G)$ ,  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ , and  $M \in \mathcal{M}(g, h)$ ,*

$$Z_M|_\gamma(v, (g, h), \tau) = \sum_{N \in \mathcal{M}(g^a h^c, g^b h^d)} \gamma_{M, N} Z_N(v, (g, h)\gamma, \tau)$$

where  $\rho(\gamma) = (\gamma_{(M, g, h), (N, g_1, h_1)})$  for  $N \in \mathcal{M}$  and  $(g_1, h_1) \in P(G)$  and we use  $\gamma_{M, N}$  to denote  $\gamma_{(M, g, h), (N, g^a h^c, g^b h^d)}$  for  $N \in \mathcal{M}(g^a h^c, g^b h^d)$  for short. That is,

$$Z_M(v, (g, h), \gamma\tau) = (c\tau + d)^{\text{wt}[v]} \sum_{N \in \mathcal{M}(g^a h^c, g^b h^d)} \gamma_{M, N} Z_N(v, (g, h)\gamma, \tau).$$

(2) *The cardinalities  $|\mathcal{M}(g, h)|$  and  $|\mathcal{M}(g^a h^c, g^b h^d)|$  are equal for any  $(g, h) \in P(G)$  and  $\gamma \in \Gamma$ . In particular, the number of inequivalent irreducible  $g$ -twisted  $V$ -modules is exactly the number of irreducible  $V$ -modules which are  $g$ -stable.*

We remark that if  $G = \{1\}$ , Part (1) was established in [Z]. Part (2) in this case is trivial.

## 4 Congruence subgroup property

We are now in a position to state the main result in this paper.

**Theorem 4.1.** *Let  $V$  be a regular, selfdual vertex operator algebra of CFT type and  $G$  a finite automorphism group of  $V$ . Then the kernel of  $\rho$  is a congruence subgroup.*

The proof of Theorem 4.1 is based on the congruence subgroup property for  $G = \{1\}$  given in [DLN]. In this case we denote  $Z_M(v, (1, 1), \tau)$  simply by  $Z_M(v, \tau)$ . Let  $\{M^0, \dots, M^p\}$  be the inequivalent irreducible  $V$ -modules. Denote the weight of  $M^i$  by  $\lambda_i$ . By Theorem 2.6,  $\lambda_i$  and  $c$  are rational numbers. Let  $s$  be the smallest positive integer such that  $s(\lambda_i - c/24)$  is an integer for all  $i$ . The following result was obtained in [DLN].

**Theorem 4.2.** *If  $V$  is a regular and selfdual and  $G = \{1\}$ , the kernel of  $\rho$  contains the congruence subgroup  $\Gamma(s)$ .*

Note that we do not need assumption that  $V$  is of CFT type in Theorem 4.2. This assumption is necessary in the current situation as Theorem 2.7 requires it.

**Lemma 4.3.** *Let  $V$  and  $G$  be as in Theorem 4.1. Suppose that  $(g, h) \in P(G)$ . Then there is a positive integer  $m_{g,h}$  such that for  $M \in \mathcal{M}(g, h)$  and  $v \in V_{[n]}$   $Z_M(v, (g, h), \tau)$  is a modular form on the congruence subgroup  $\Gamma(m_{g,h})$  of weight  $n$ .*

*Proof.* Let  $H$  be the subgroup of  $G$  generated by  $g, h$ . Then  $H$  is a finite abelian group. By Theorem 2.7,  $V^H$  is a regular vertex operator algebra. We expect to apply Theorem 4.2 to vertex operator algebra  $V^H$ . So we need to verify that  $V^H$  is selfdual.

We know from [DM] and [DLM1] that  $V$  has the following decomposition

$$V = \bigoplus_{\mu \in \hat{H}} V^\mu$$

where  $\hat{H}$  is the set of irreducible characters of  $H$  and  $V^\mu$  is the subspace of  $V$  corresponding to the character  $\mu$ . Since  $V$  is selfdual, it is easy to see that the dual of  $V^\mu$  is  $V^{\mu^{-1}}$ . In particular, if  $\mu = 1$  is the trivial character of  $H$ ,  $V^H = V^1$  is selfdual.

From Theorem 4.2 there is a positive integer  $m_{g,h}$  such that  $Z_W(v, \tau)$  is a modular form of weight  $n$  on congruence subgroup  $\Gamma(m_{g,h})$  for any irreducible  $V^H$ -module  $W$  and any  $v \in V_{[n]}^H$ . The main idea is to express  $Z_M(v, (g, h), \tau)$  as a linear combination of  $Z_W(v, \tau)$  for some irreducible  $V^H$ -modules  $W$ .

Since  $V^H$  is rational, it follows from a result in [DRX] that any irreducible  $V^H$ -module occurs in an irreducible  $k$ -twisted  $V$ -module for some  $k \in H$ . Let  $H_M = \{k \in H \mid M \circ k \cong M\}$  be the stabilizer of  $M$  in  $H$ . Then  $H_M$  acts on  $M$  projectively. That is, there exists a 2-cocycle  $\alpha \in H^2(H_M, S^1)$  such that  $\varphi(k_1)\varphi(k_2) = \alpha(k_1, k_2)\varphi(k_1k_2)$  for  $k_i \in H_M$  where  $S^1$  is the group of unit circle. Then the twisted group algebra  $\mathbb{C}^\alpha[H_M]$  is a finite dimensional semisimple associative algebra. Denote by  $\Lambda_M$  the set of irreducible characters of  $\mathbb{C}^\alpha[H_M]$ . For each  $\lambda \in \Lambda_M$  let  $W_\lambda$  be the corresponding irreducible  $\mathbb{C}^\alpha[H_M]$ -module. This leads to the following decomposition

$$M = \bigoplus_{\lambda \in \Lambda_M} W_\lambda \otimes M_\lambda$$

where  $M_\lambda$  is the multiplicity space of  $W_\lambda$  in  $M$ . From [DM], [DLM1], [DY], [MT],  $M_\lambda$  is an irreducible  $V^H$ -module. In particular,  $M$  is a completely reducible  $V^H$ -module.

We can assume that  $\varphi(h)$  has finite order  $K$ . Then  $M = \bigoplus_{j=0}^{K-1} M_j$  where  $M_j$  is the eigenspace of  $\varphi(h)$  with eigenvalue  $e^{2\pi i j/K}$ . Using the identity

$$\varphi(h)Y_M(v, z)\varphi(h)^{-1} = Y_M(hv, z)$$

we see that  $\varphi(h)$  and  $Y_M(v, z)$  commute for  $v \in V^H$ . This implies that each  $M_j$  is a  $V^H$ -submodule of  $M$  and is a direct sum of irreducible  $V^H$ -modules  $M_\lambda$  for  $\lambda \in \Lambda_M$ . It follows from the definition that

$$Z_M(v, (g, h), \tau) = \sum_{j=0}^{K-1} e^{2\pi i j/K} \text{tr}_{M_j} o(v) q^{L(0)-c/24}.$$

Clearly, each  $Z_{M_j}(v, \tau) = \text{tr}_{M_j} o(v) q^{L(0)-c/24}$  is a modular form on  $\Gamma(m_{g,h})$  of weight  $n$  for any  $v \in V_{[n]}^H$ . So is  $Z_M(v, (g, h), \tau)$ .  $\square$

**Proof of Theorem 4.1:** Let  $m$  be the least common multiple of all  $m_{g,h}$  for  $(g, h) \in P(G)$ . Then for any  $v \in V_{[n]}^G$  and  $M \in \mathcal{M}$ ,  $Z_M(v, (g, h), \tau)$  is a modular form on  $\Gamma(m)$  of weight  $n$ . That is, the kernel of  $\rho$  contains the congruence subgroup  $\Gamma(m)$ . The proof is complete.

Although we do not know  $V^G$  is rational or  $C_2$ -cofinite when  $G$  is not abelian, but we still have:

**Corollary 4.4.** *If  $V$  is a regular, selfdual vertex operator algebra of CFT type and  $G$  is any finite automorphism group of  $V$  then for any irreducible  $V^G$ -module  $N$  occurring in an irreducible  $g$ -twisted  $V$ -module  $M$  with  $g \in G$ ,  $Z_N(v, \tau)$  is a modular form of weight  $n$  on the congruence subgroup  $\Gamma(m)$  where  $m$  is the same as before and  $v \in V_{[n]}^G$ . In particular, the  $q$ -character  $\chi_N(\tau)$  is a modular function on  $\Gamma(m)$ .*

*Proof.* First, we recall the irreducible  $V^G$ -module occurring in an irreducible  $g$ -twisted  $V$ -module  $M$  from [MT] and [DRX]. The above discussion tells us that  $h \mapsto \varphi(h)$  gives a projective representation of  $G_M$  on  $M$ . Let  $\alpha_M \in H^2(G_M, S^1)$  be the corresponding 2-cocycle. Then

$$\begin{aligned}\varphi(h)\varphi(k) &= \alpha_M(h, k)\varphi(hk), \\ \varphi(h)Y_M(v, z)\varphi(h)^{-1} &= Y_M(hv, z)\end{aligned}$$

for  $h, k \in G_M$  and  $v \in V$ . As before,  $M$  is a module for the twisted group algebra  $\mathbb{C}^{\alpha_M}[G_M]$  and we have decomposition

$$M = \bigoplus_{\lambda \in \Lambda_{G_M, \alpha_M}} W_\lambda \otimes M_\lambda$$

where  $\Lambda_{G_M, \alpha_M}$  is the set of irreducible characters of  $\mathbb{C}^{\alpha_M}[G_M]$ ,  $W_\lambda$  is the irreducible  $\mathbb{C}^{\alpha_M}[G_M]$ -module with character  $\lambda$  and  $M_\lambda$  is the multiplicity of  $W_\lambda$  in  $M$ .

It follows from [MT] and [DRX] that each  $M_\lambda$  is an irreducible  $V^G$ -module. Moreover,  $M_\lambda$  and  $M_\mu$  are not isomorphic if  $\lambda, \mu$  are different. Let  $\lambda, \mu$  be two irreducible characters of  $\mathbb{C}^{\alpha_M}[G_M]$ . The following orthogonal relation holds:

$$\frac{1}{|G_M|} \sum_{h \in G_M} \lambda(h) \overline{\mu(h)} = \delta_{\lambda, \mu}$$

where  $\overline{\mu(h)}$  is the complex conjugate of  $\mu(h)$  by Lemma 4.3 of [DRX]. We can now assume that  $N = M_\lambda$  for some  $\lambda \in \Lambda_{G_M, \alpha_M}$ . Then we can easily get

$$Z_{M_\lambda}(v, \tau) = \frac{1}{|G_M|} \sum_{h \in G_M} \overline{\lambda(h)} Z_M(v, (g, h), \tau).$$

By Theorem 4.1, each  $Z_M(v, (g, h), \tau)$  is a modular form on  $\Gamma(m)$ . The result follows immediately.  $\square$

From Corollary 4.4 we know that  $q$ -character  $\chi_{V^G}(\tau)$  of  $V^G$  is a modular function. This suggests the following conjecture:



**Conjecture 4.5.** *Let  $U$  be a simple vertex operator algebra. Then*

*(1)  $U$  is rational if and only if the  $q$ -character of each irreducible  $U$ -module is a modular function on a same congruence subgroup.*

*(2)  $U$  is rational if and only if the  $q$ -character of  $U$  is a modular function on a congruence subgroup.*

Clearly, Conjecture (2) implies (1). These conjectures tell us that the modularity of the  $q$ -character of a vertex operator algebra is unique for rational vertex operator algebras. If this conjecture is true, it would imply that  $V^G$  is rational for any finite automorphism group  $G$  of  $V$ .

The following conjecture is a converse of Zhu's result [Z].

**Conjecture 4.6.** *Let  $U$  be a simple,  $C_2$ -cofinite vertex operator algebra. Let  $M^0, \dots, M^p$  be the irreducible  $U$ -modules. Then  $U$  is rational if and only if the space spanned by  $\{Z_{M^i}(v, \tau) | i = 0, \dots, p\}$  affords a representation of the modular group  $\Gamma$ .*

## 5 Holomorphic orbifolds

Recall that vertex operator algebra  $V$  is called holomorphic if it is rational and has a unique irreducible module, namely  $V$  itself. Let  $V$  be a  $C_2$ -cofinite, holomorphic vertex operator algebra of CFT type and  $G$  a finite automorphism group of  $V$ . Then for each  $g \in G$  there is a unique irreducible  $g$ -twisted  $V$ -module  $V(g)$  by Theorem 3.1. For short we write  $Z(v, (g, h), \tau)$  for  $Z_{V(g)}(v, (g, h), \tau)$  for  $(g, h) \in P(G)$  and  $Z(g, h, \tau)$  for  $Z(\mathbf{1}, (g, h), \tau)$ .

**Proposition 5.1.** *The following hold:*

*(1) For any  $k \in G$ ,  $Z(v, (kgk^{-1}, khk^{-1}), \tau) = Z(v, (g, h), \tau)$ .*

*(2) For any  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ , and  $(g, h) \in P(G)$  and  $v \in V_{[n]}^G$*

$$Z(v, (g, h), \gamma\tau) = (c\tau + d)^{\text{wt}[v]} \gamma_{(g, h), (g^a h^c, g^b h^d)} Z(v, (g^a h^c, g^b h^d), \tau).$$

*(3)  $Z(v, (g, h), \tau)$  is a modular form of weight  $n$  on the congruence subgroup  $\Gamma(m)$  where  $(g, h) \in P(G)$ ,  $v \in V_{[n]}^G$ , and  $m$  is given in Theorem 4.1.*

*(4) The coefficient of each power of  $q$  in  $Z(g, h, \tau)$  defines a projective character of  $C_G(g)$ .*

*In particular, (1)-(4) hold for  $Z(g, h, \tau)$ .*

*Proof.* Part (3) is a special case of Theorem 4.1. The rest was given in [DLM4].  $\square$

In the case that  $V$  is holomorphic and  $h$  is a power of  $g$ , the modularity of  $Z(g, h, \tau)$  was obtained previously in [EMS].

We know from [D], [DGH] that moonshine vertex operator algebra  $V^\natural$  [FLM] is a  $C_2$ -cofinite, holomorphic vertex operator algebra of CFT type. Moreover, the automorphism group of  $V^\natural$  is the monster simple group [FLM]. In this case, the functions  $Z(g, h, \tau)$  are the candidates in the generalized moonshine conjecture [N], [DLM4]. Parts (1), (2), (4) were proved to hold in [DLM4]. What new here is that each generalized McKay-Thompson

series  $Z(g, h, \tau)$  is a modular function over a congruence subgroup  $\Gamma(m)$ . So unfinished business in proving the generalized moonshine conjecture is to extend the group  $\Gamma(m)$  to a genus zero subgroup of  $SL(2, \mathbb{R})$ .

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